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LETTER TO THE EDITOR

Regular random fractals and the n-parameter model†

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Abstract. Two models of fractal objects are presented. The first model represents a new class of fractals which is intermediate to statistical fractals and exact fractals in that each configuration of the random ensemble has an exact (and adjustable) fractal dimension. The second model has a continuously adjustable backbone dimension and can be used to directly generate conducting backbones.

Currently, volume fractals fall into two completely distinct classes: exact fractals such as the Sierpinski gasket and Halvin carpet (see Mandelbrot 1977), and statistical fractals such as percolation clusters (Stauffer 1979) and Witten–Sander aggregates (Witten and Sander 1981, 1982). In the first class the fractal dimension is a well defined exact property of the cluster while in the second class the fractal dimension is defined only for the ensemble. In this letter we demonstrate a third class of fractals, which we call regular random fractals (RRFS), in which individual members of the random ensemble have well defined exact scaling properties. Also the fractal dimension, d_f, of this new model can be adjusted over a countably infinite set of values which span the physical limits $1 \leq d_f \leq d$, where d is the dimension of space.

We also present a second, related model in which fluctuations are admitted to the ensemble. In this n-parameter model the number of parameters can be made arbitrarily large. Computer simulations are presented for the $n = 2$ model in two dimensions and it is found that in this case both the fractal dimension and the fractal dimension of the conducting backbone (Shlifer *et al* 1979), d_{BB}, can be adjusted over a set of uncountably infinite values. Simulations show that the upper limit of the conducting backbone dimension is $d_{BB} = d_f$, which corresponds to the direct generation of fully conducting fractals. These fully conducting fractals have a 'link-blob' morphology and may be a suitable model for the percolation backbone if the fractal dimension is suitably adjusted.

The method of constructing RRFS proceeds iteratively and involves three steps: partitioning, deletion and renormalisation. The first generation is obtained from the zeroth generation, the d-dimensional cube, in the following way: (1) the d-dimensional cube is partitioned into p^d equal-volume d-dimensional cubes by slicing the original hypercube $p-1$ times on each side; (2) q of the resultant p^d hypercubes are randomly deleted such that the remaining hypercubes are connected, where two cubes are considered connected if they share a $(d-1)$-dimensional edge; (3) the resultant cluster is renormalised by the factor p in each dimension. The process is continued *ad infinitum*

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(while always maintaining local connectivity), and a random cluster of fractal dimension

$$d_f = \log(p^d - q)/\log(p)$$

emerges from the iteration, where p and q are integers such that $0 \leq q \leq p^d - p$ (i.e. $d \geq d_f \geq 1$). Here $p^d - q$ is the ratio of the masses of the ith and $(i+1)$th generations, and p is the factor by which the length scale is transformed.

The stepwise generation of a $d = 2$ regular random fractal is shown in figure 1 for the case $p = 2$, $q = 1$. In this case $d_f = \log(3)/\log(2) = 1.585$, which is just the fractal dimension of the Sierpinski gasket. In fact, the Sierpinski gasket is a member of this regular random ensemble, and a Havlin carpet, with fractal dimension $\log(8)/\log(3)$, is a member of the $d = 2$, $p = 3$, $q = 1$ ensemble. Many other familiar fractals emerge from these ensembles as well.

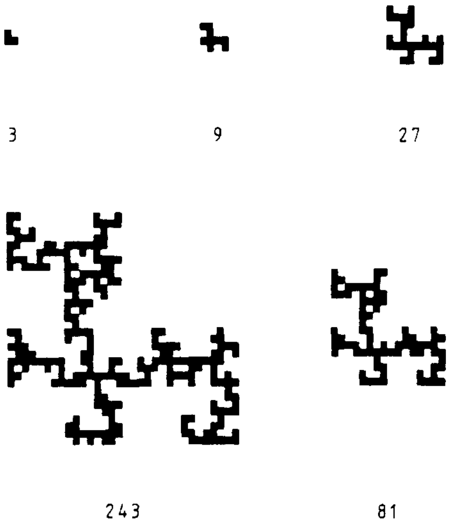

Figure 1. The first five generations of a regular random fractal ($d_f = \log(3)/\log(2)$) are shown clockwise from the top left. The fifth generation cluster has a mass of 243 sites—the zeroth generation, of mass 1, is not shown.

Each of the regular random fractals can be transformed into an $n = p^d = p$ parameter continuous random fractal model. In this model the parameter q is allowed to fluctuate, and is therefore only statistically defined. Let f_j be the probability, after partitioning, of deleting j of p^d hypercubes. Then the mean value of q is

$$\langle q \rangle = \sum_0^n j f_j$$

where the f_j are normalised by $\sum f_j = 1$. The fractal dimension of the ensemble is then just $\log(p^d - \langle q \rangle)/\log(p)$. This class of many-parameter fractals is very complex—in general there are n parameters, which leaves $n - 1$ degrees of freedom after the fractal dimension is specified. What properties do these $n - 1$ parameters control? Are the additional parameters irrelevant insofar as exponents are concerned? To get some idea of the behaviour of these n-parameter fractals, we have chosen to study the simplest example, where $d = p = 2$.

In this two-parameter model it is useful to define parameters which depend on the f_j in a nonlinear fashion. The first parameter is simply $\log(p^d - \langle q\rangle)/\log(p)$, the fractal dimension. The second parameter, conveniently defined to lie on [0, 1], is

$$\alpha = (f_2 - f_2^{\min})/(f_2^{\max} - f_2^{\min})$$

where $f_2^{\min}$ is the minimum possible probability of deleting two squares consistent with obtaining a given fractal dimension, and $f_2^{\max}$ is the corresponding maximum. In terms of $\langle q\rangle$, these are

$$f_2^{\max} = \langle q\rangle/2$$

$$f_2^{\min} = \begin{cases} 0 & \langle q\rangle \leqslant 1 \\ \langle q\rangle - 1 & \langle q\rangle > 1. \end{cases}$$

The striking variation of the geometry of clusters produced by this model can be seen in figure 2, where clusters are shown for $d_f = 1.585$, 1.4 and 1.2. The clusters on the left-hand side were made with $\alpha = 0$ and the clusters on the right were made with $\alpha = 0.8$. It is immediately apparent that $\alpha = 0$ clusters have a homogeneous appearance—the neighbourhood of any site is roughly equivalent—whereas the $\alpha = 0.8$ clusters have an extremely heterogeneous appearance, with both 'urban' and 'rural' areas. Also, the $\alpha = 0$ fractals have a highly branched structure, with very few circuits, whereas the $\alpha = 0.8$ fractals have many circuits (especially those of length 4) and very few branches.

Although many, and perhaps all, of the exponents which are now used to describe fractals are dependent on α, the most apparent difference between these clusters is the degree of branching, which can be quantified by the fractal dimension of their conducting backbone. This conducting backbone may be defined as the sites which carry current when electrodes are placed at the topologically maximally separated sites of a cluster, or may also be thought of as the intersection of all self-avoiding walks between these sites (Shlifer *et al* 1979). Since we obtained the conducting backbone by the method of burning (Stauffer 1985, Herrmann *et al* 1984), we have investigated the fractal dimension of the elastic backbone (Herrmann *et al* 1984) d^{EB} as well. The elastic backbone is simply defined as the union of all minimal paths between these maximally separated sites, where a minimal path is the topologically shortest walk, on the fractal, from one site to another.

In figure 3 the dependence of the fractal dimension of the conducting backbone on α is shown for $d_f = 1.2$, 1.4, 1.585 and 1.7. To obtain these data simulations were carried through seven generations (lattice size 128×128) and averages were taken over 1200 clusters. It is seen that d^{BB} is very dependent on α for small d_f, but for $d_f = 1.7$ the dependence is very small. In fact, for $d_f = 1.8$ no α dependence could be observed for d^{BB}. The most significant feature of figure 3, however, is the large α behaviour of d^{BB}. In the limit $\alpha = 1$ it can be seen that all four curves extrapolate to 1, so that $d^{\mathrm{BB}} = d_f$. This $\alpha = 1$ limit corresponds to the direct generation of conducting backbones of arbitrary connectivity. For example, an analytical model for the percolation backbone may arise from setting $d_f = 1.60$ (Herrmann *et al* 1984) and $\alpha = 1$. Of course, it remains to be seen if other properties of the percolation backbone scale with cluster radius in the same way, but if this is the case then this model will represent the first analytical representation of the percolation backbone.

The dependence of the elastic and conducting backbone dimensions on the cluster fractal dimension is shown in figure 4 for the homogeneous limit, $\alpha = 0$. Apparently, d^{BB} depends linearly on the fractal dimension since the conducting backbone data

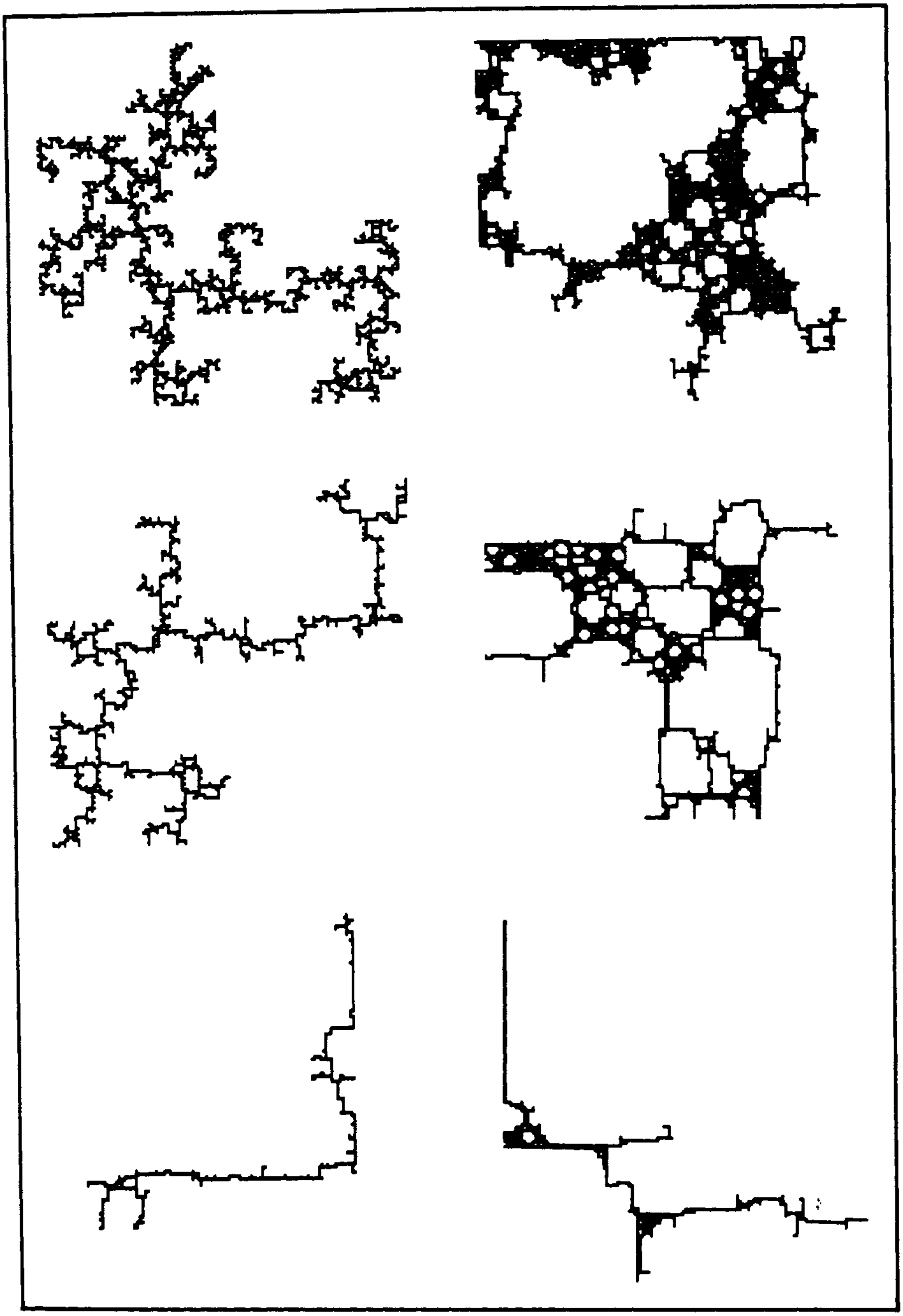

Figure 2. The variation in fractal structure with d_f and α is shown. From the top down, the fractal dimension is 1.585, 1.4 and 1.2. From left to right $\alpha = 0, 0.8$.

extrapolate nicely to the necessary limit $d^{BB} = d_f$ at $d_f = d$. On the other hand, the data for the elastic backbone do not appear to extrapolate linearly to the required limit $d^{EB} = 1$ at $d_f = d$, indicating a nonlinear dependence of the elastic backbone

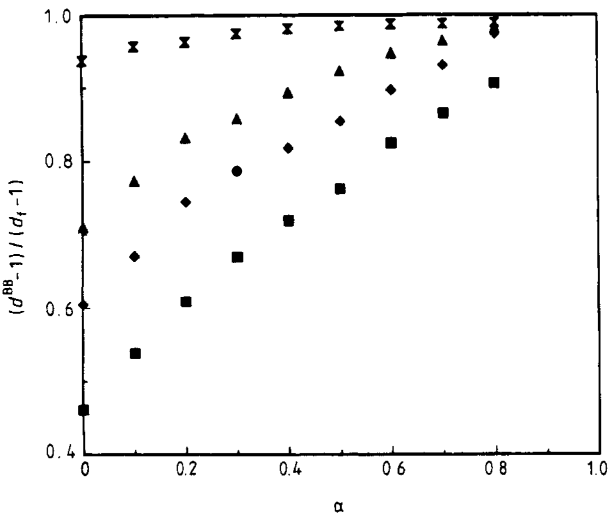

Figure 3. The dependence of the fractal dimension of the conducting backbone on α is shown on relative axes for clusters of fractal dimensions 1.2 (■), 1.4 (◆), 1.585 (▲) and 1.7 (X). Data were obtained by averaging over 1200 seventh-generation (128 × 128) clusters.

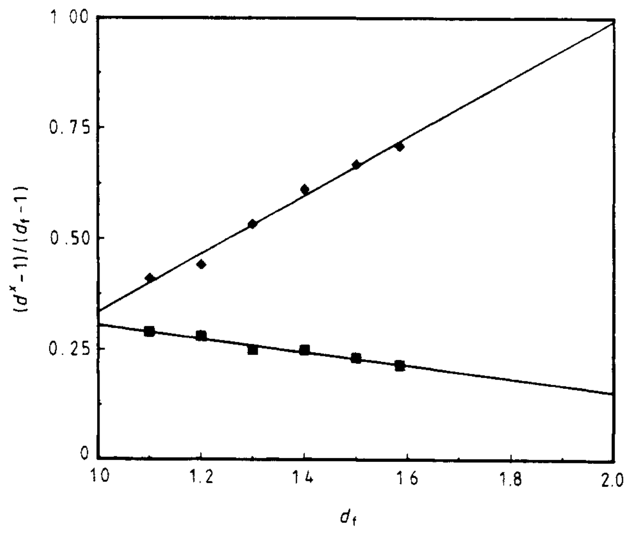

Figure 4. The fractal dimensions of the conducting (◆) and elastic (■) backbones are shown as functions of d_f for $\alpha = 0$. Within the errors in the data these backbones have the same fractal dimension for small d_f, indicating that these low-dimensional fractals have very few circuits.

dimension on the fractal dimension for large d_f. In the limit of small fractal dimension $d^{BB} = d^{EB} = 1 + k(d_f - 1)$, where $k \sim 0.3$. This equality of d^{BB} and d^{EB} is due to a dearth of circuits in low fractal dimension clusters—in purely branched structures the elastic and conducting backbones are identical. Finally, the dependence of the elastic backbone fractal dimension on α can be seen in figure 5 for $d_f = 1.585$. Since d^{EB} is very close to 1, the errors in these data are fairly large, but qualitatively it would appear that in the heterogeneous link-blob limit, the elastic backbone fractal dimension approaches 1.

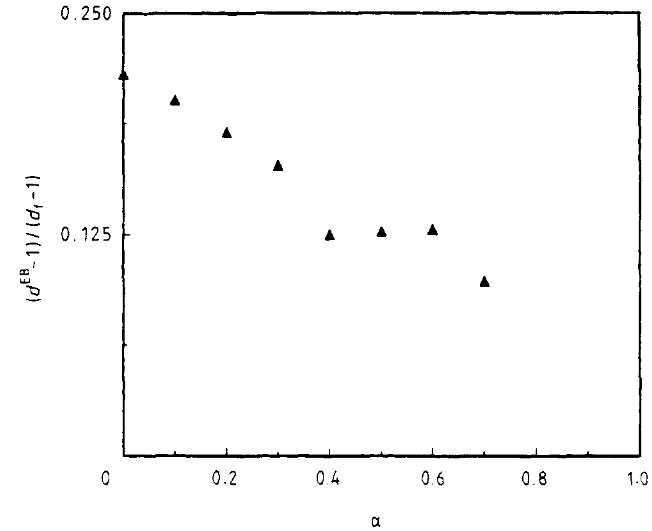

Figure 5. The fractal dimension of the elastic backbone is shown as a function of α for $d_f = 1.585$. Due to the low dimensionality of the elastic backbone the scatter in the data is fairly large.

In summary, we have demonstrated two new classes of fractals: regular random fractals and the n-parameter model. The RRFS are characterised by non-fluctuating ensembles of discretely variable fractal dimension, so that the fractal dimension is a property of both the ensemble and an individual cluster. On the other hand, the n-parameter model is permitted to fluctuate, so that the fractal dimension is defined only for the ensemble. It is found that even the simplest ($n = d = 2$) n-parameter model has a rich behaviour, with both continuously variable fractal and backbone dimensions. Also, when the parameter $\alpha = 1$, the two-parameter model produces backbone fractals. This suggests the possibility of using the $\alpha = 1$, $d_f = 1.60$ as an analytical model for two-dimensional percolation backbones.

Finally, it is interesting to conjecture about the potential of the n-parameter model in the limit of large n. If there is but a finite number of exponents, m, which describe fractal objects, then the parameter space for $n > m$ must consist of a k-dimensional subspace ($k < m$) which contains all possible scaling properties of this model. If $k = m$ then any fractal object can be mapped onto an appropriate version of this model.

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